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Coherent states on the circle

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Abstract. A careful study of the physical properties of a family of coherent states on the circle, introduced some years ago by de Bièvre and González (in 1992 Semiclassical behaviour of the Weyl correspondence on the circle *Group Theoretical Methods in Physics* vol I (Madrid: Ciemat)), is carried out. They were obtained from the Weyl–Heisenberg coherent states in $L^2(\mathbb{R})$ by means of the Weil–Brezin–Zak transformation, they are labelled by the points of the cylinder $S^1 \times \mathbb{R}$, and they provide a realization of $L^2(S^1)$ by entire functions (similar to the well known Fock–Bargmann construction). In particular, we compute the expectation values of the position and momentum operators on the circle and we discuss the Heisenberg uncertainty relation.

1. Introduction

This paper is devoted to the study of the physical properties of a family of coherent states (CS) defined on the circle (i.e. belonging to $L^2(S^1)$) and labelled by the points of the cylinder. These CS were introduced by de Bièvre and González in [DG 92, DG 93], where they were simply sketched. Here we study them more deeply. Our aim is to contribute to the development of the quantum theory on periodic phase spaces. Among these phase spaces we pay particular attention to the cylinder because of their relation with physical systems with periodic motion and their non-trivial topology. Moreover, the quantum formalism on the cylinder is far from being completely understood.

It has been proven that families of CS are relevant in the study of many quantum systems [KS 85, PE 86], but this formalism presents some difficulties when one wishes to apply it to the cylinder. For instance, the cylinder can be seen as a coadjoint orbit of the Euclidean group of the plane but, in a strict sense, the Perelomov method [PE 86] for constructing CS with this group does not work (de Bièvre [DB 89] and Isham and Klauder [IK 91] have demonstrated two different ways in which to avoid this problem). Nevertheless, the CS introduced here are not obtained by any of these procedures, but by decomposition of the standard Weyl–Heisenberg CS on $\mathbb{R}$. The machinery to carry out such a decomposition is the Weil–Brezin–Zak (WBZ) transform [JA 82, FO 89], which was originally used for the study of periodic potentials [ZA 68, RS 78]. This WBZ transform relates the quantum formalisms on the plane and on the cylinder (or on the torus considered as phase space [BB 96]). Roughly speaking, this procedure maps functions of one variable on quasiperiodic functions of two variables by a generalization of the Bloch functions.

As an application, these CS can be used to study a quantum particle on the circle as has been done recently by Kowalski *et al* [KR 96]. Although they assume to use CS different

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from ours and to have obtained a better approach to this problem, it is easy to prove that their CS are a particular case of the CS used here, which shows the wider generality of our approach.

The paper is organized as follows. In section 2 we review the main properties of the WBZ transform that plays a central role in our work. Section 3 is devoted to the CS on the circle, which are obtained by decomposition of the standard Weyl–Heisenberg CS on $\mathbb{R}$ (i.e. CS belonging to $L^2(\mathbb{R})$); in other words, the CS on the circle are the image of the Weyl–Heisenberg CS by the WBZ transform. These CS on the circle provide a realization of the space $L^2(S^1)$ in terms of entire functions as shown in section 4, in analogy with the Fock–Bargmann representation of $L^2(\mathbb{R})$ provided by the Weyl–Heisenberg CS. Part of the results of sections 3 and 4 have been published in [DG 92]. Section 5 presents a generalization of the CS on the circle to an n-dimensional torus $\mathbb{T}^n$, thus we will obtain a family of CS in $L^2(\mathbb{T}^n)$. The physical properties are studied in section 6, paying special attention to the expectation values of the position and the (angular) momentum operators, and to the Heisenberg uncertainty relation. The last section is devoted to proving that the CS of [KR 96] agree with our CS for the particular parameter values that characterize theirs, and to present some conclusions.

2. The Weil–Brezin–Zak transform

It is a well known fact that $L^2(\mathbb{R})$ is isomorphic to $L^2(S^1 \times S^{1*})$, where S^{1*} is the dual space of S^1. This result has been used, for instance, in solid state theory to construct the Bloch functions [ZA 68, RS 78], as well as for quantum description of periodic variables [ZA 69]. In this context, we call the WBZ transform T to the unitary map from $L^2(\mathbb{R})$ to $L^2(S^1 \times S^{1*})$ [JA 82, FO 89]. If we identify S^1 with the interval $[0, a)$ and S^{1*} with $[0, 2\pi/a)$, then T is explicitly given by

$$(T\psi)(q,k) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}nak}\psi(q-na) \tag{2.1}$$

for $\psi \in L^2(\mathbb{R}), q \in S^1$ and $k \in S^{1*}$. Conversely,

$$\psi(q-na) = \frac{a}{2\pi}\int_0^{2\pi/a} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}nak}(T\psi)(q,k) \qquad q \in S^1, n \in \mathbb{Z}. \tag{2.2}$$

In this way, the functions $T\psi$ are periodic in k and quasiperiodic in q,

$$(T\psi)\left(q+na, k+m\frac{2\pi}{a}\right) = \mathrm{e}^{\mathrm{i}nak}(T\psi)(q,k) \qquad n, m \in \mathbb{Z}. \tag{2.3}$$

Note that if we fix a value of k (as we are going to do from now on) the operator given by (2.1) is a projection onto $L^2(S^1)$, which we denote by $T^{(k)}$, and we get the so-called constant fibre direct integral decomposition [RS 78],

$$L^2(\mathbb{R}) \cong \int_{S^{1*}}^{\oplus} \mathrm{d}k\, L^2(S^1). \tag{2.4}$$

In this case, we will frequently use the notation $T^{(k)}\psi = \psi^{(k)}$, and we will say that $\psi^{(k)}$ is obtained by decomposition of ψ.

3. Coherent states on the circle

In this section we show that a family of CS in $L^2(S^1)$ can be constructed by decomposition of the standard Weyl–Heisenberg CS in $L^2(\mathbb{R})$. The latter are given as an orbit under the Weyl–Heisenberg group [KS 85, PE 86]:

$$\eta_{y,p}(x) := \exp\left(\frac{\mathrm{i}}{\hbar}(pQ - yP)\right)\eta_0(x) = \exp\left(\frac{\mathrm{i}}{\hbar}p\left(x - \frac{y}{2}\right)\right)\eta_0(x-y) \tag{3.1}$$

where $x, y, p \in \mathbb{R}$ and $\eta_0 \in L^2(\mathbb{R})$ is a fiducial state, which is usually chosen to be a normalized Gaussian:

$$\eta_0(x) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\omega}{2\hbar}x^2\right). \tag{3.2}$$

Now, we can use (2.1) to construct the functions $\eta_{y,p}^{(k)} \in L^2(S^1)$ (or $|y, p; k\rangle$ in Dirac's notation) and it is natural to ask if, for each value of k, this set of functions will also be a set of CS, labelled by suitable values of y and p. The answer is positive, according to the generalized definition of CS given in [KS 85]: simply, a family of states depending continuously on a set of labels and fulfilling a resolution of the unity. They are not constructed by Perelomov's method [PE 86], as an orbit under a Lie group representation. Actually, we have here a non-trivial example of the 'reproducing triplets' introduced in [AA 91].

Theorem 3.1. For each $k \in S^{1*}$, the family $\{\eta_{q,p}^{(k)} \equiv |q, p; k\rangle | (q, p) \in S^1 \times \mathbb{R}\}$, where $\eta_{q,p}$ is given by (3.1) with $\|\eta_0\| = 1$, is a set of coherent states in $L^2(S^1)$; i.e. they verify the following resolution of unity:

$$\frac{1}{2\pi\hbar}\int_0^a \mathrm{d}q \int_{-\infty}^{\infty}\mathrm{d}p\, |q, p; k\rangle\langle q, p; k| = I. \tag{3.3}$$

The proof consists of a simple calculation, using definitions (2.1) and (3.1) [GO 96]. If we choose η_0 according to (3.2), these CS take the form

$$\eta_{q,p}^{(k)}(q') = \left(\frac{\omega}{\pi\hbar}\right)^{1/4}\exp\left(\frac{\mathrm{i}}{2\omega\hbar}pz^*\right)\exp\left(-\frac{1}{2\omega\hbar}(z^* - \omega q')^2\right) \times\theta\left(\mathrm{i}\frac{a}{2\hbar}(z^* - \omega q' - \mathrm{i}k\hbar); \rho_1\right) \tag{3.4}$$

where $z^* = \omega q + \mathrm{i}p$, $\rho_1 = \exp(-\frac{a^2\omega}{2\hbar})$, and $\theta(z; \rho) = \sum_{n=-\infty}^{\infty}\rho^{n^2}\mathrm{e}^{2\mathrm{i}nz}$, $|\rho| < 1$, is the Theta function (sometimes denoted by θ_3) [WW 27, AS 72, ER 81, MU 83].

As a corollary to the preceding theorem, we present a typical property of every set of CS [KS 85].

Corollary 3.2. The mapping $W^{(k)} : L^2(S^1) \to L^2(S^1 \times \mathbb{R})$ given by

$$(W^{(k)}\varphi)(q, p) = \langle q, p; k|\varphi\rangle \tag{3.5}$$

is an isometry, and $W^{(k)}\big(L^2(S^1)\big)$ is a reproducing kernel space, with kernel

$$\frac{1}{2\pi\hbar}\langle q', p'; k|q, p; k\rangle.$$

To compute this kernel, let us consider the orthonormal basis in $L^2(S^1)$:

$$\left\{|n; k\rangle \equiv \frac{1}{\sqrt{a}}\exp\mathrm{i}\left(\frac{2\pi}{a}n + k\right)q \qquad n \in \mathbb{Z}, k \in S^{1*} \text{ fixed}\right\}. \tag{3.6}$$

Then we can write

$$|q,\,p;k\rangle = \sum_{n=-\infty}^{\infty} c_n^{(q,p;k)}|n;k\rangle \tag{3.7}$$

where the coefficients are

$$c_n^{(q,p;k)} = \sqrt{\frac{2\pi\hbar}{a}}\mathrm{e}^{\mathrm{i}[p/(2\hbar)-(2\pi n/a+k)]q}\,\widetilde{\eta_0}\left(\left(\frac{2\pi}{a}n+k\right)\hbar - p\right) \tag{3.8}$$

$\widetilde{\eta_0}$ being the Fourier transform of η_0. Now, using (3.2) and (3.8), we easily obtain

$$\begin{aligned}\langle q',\,p';k|q,\,p;k\rangle &= \sum_{n=-\infty}^{\infty}\langle q',\,p';k|n;k\rangle\langle n;k|q,\,p;k\rangle = \sum_{n=-\infty}^{\infty} c_n^{(q',p';k)*}c_n^{(q,p;k)}\\ &= \frac{2}{a}\sqrt{\frac{\pi\hbar}{\omega}}\mathrm{e}^{\mathrm{i}k(q'-q)}\mathrm{e}^{\mathrm{i}(qp-q'p')/2\hbar}\mathrm{e}^{-[(\hbar k-p)^2+(\hbar k-p')^2]/2\omega\hbar}\\ &\quad\times\theta\left(\frac{\pi}{a}\left[(q'-q)+\frac{\mathrm{i}}{\omega}(2\hbar k-p-p')\right];\rho_2\right)\end{aligned} \tag{3.9}$$

where $\rho_2 = \exp(-\frac{4\pi^2\hbar}{\omega a^2})$.

Note that these CS are not normalized. It follows immediately from (3.9) that

$$\langle q,\,p;k|q,\,p;k\rangle = \frac{2}{a}\sqrt{\frac{\pi\hbar}{\omega}}\mathrm{e}^{-(\hbar k-p)^2/(\omega\hbar)}\theta\left(\mathrm{i}\frac{2\pi}{\omega a}(\hbar k-p);\rho_2\right). \tag{3.10}$$

Taking into account the identity

$$\theta(z;\rho_2) = \frac{a}{2}\sqrt{\frac{\omega}{\pi\hbar}}\mathrm{e}^{-\omega a^2z^2/(4\pi^2\hbar)}\theta\left(-\mathrm{i}\frac{\omega a^2}{4\pi\hbar}z;\rho_1^{1/2}\right) \tag{3.11}$$

which is easily deduced from the so-called functional equation of θ [ER 81, MU 83], we also obtain the expression

$$\langle q,\,p;k|q,\,p;k\rangle = \theta\left(\frac{a}{2\hbar}(\hbar k-p);\rho_1^{1/2}\right). \tag{3.12}$$

4. A realization of $L^2(S^1)$ by analytic functions

Let us consider again the isometry $W^{(k)}$ given by (3.5). If we define the new mapping

$$(B^{(k)}\varphi)(z) = \exp\left(\frac{\mathrm{i}}{2\omega\hbar}pz\right)(W^{(k)}\varphi)(q,\,p) \tag{4.1a}$$

$$= \exp\left(\frac{\mathrm{i}}{2\omega\hbar}pz\right)\langle q,\,p;k|\varphi\rangle \qquad \varphi\in L^2(S^1) \tag{4.1b}$$

with $z = \omega q - \mathrm{i}p$, then $B^{(k)}\varphi$ is an analytic function on $S^1 + \mathrm{i}\mathbb{R}$ (because $\theta(z;\rho)$ is an entire function of z). This suggests that we should search for a representation of $L^2(S^1)$ by entire functions, similar to the standard Fock–Bargmann representation of $L^2(\mathbb{R})$ [BA 61, PE 86, FO 89]. In this context, it is quite natural to define a new set of CS, labelled by $z^* = \omega q + \mathrm{i}p$, by

$$|z^*;k\rangle = \exp\left(-\frac{\mathrm{i}}{2\omega\hbar}pz^*\right)|q,\,p;k\rangle \tag{4.2a}$$

$$\eta_{z^*}^{(k)}(q') = \left(\frac{\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{(z^*-\omega q')^2}{2\omega\hbar}\right)\theta\left(\mathrm{i}\frac{a}{2\hbar}(z^*-\omega q'-\mathrm{i}k\hbar);\rho_1\right) \tag{4.2b}$$

such that we simply have

$$(B^{(k)}\varphi)(z) = \langle z; k|\varphi\rangle \qquad \varphi \in L^2(S^1). \tag{4.3}$$

Note that we write $|z^*; k\rangle^\dagger = \langle z; k|$. Since $|z^* + \omega a; k\rangle = \mathrm{e}^{-\mathrm{i}ak}|z^*; k\rangle$ (which is easy to check) we can extend $(B^{(k)}\varphi)(z)$ to the whole of $\mathbb{C}$, so obtaining an entire function of z, $\forall\varphi \in L^2(S^1)$. Moreover, the CS $|z^*; k\rangle$ fulfil the resolution of the unity:

$$\frac{1}{2\pi\hbar}\int_0^a \mathrm{d}q \int_{-\infty}^{\infty} \mathrm{d}p\, \mathrm{e}^{-p^2/(\omega\hbar)}|z^*; k\rangle\langle z; k| = I. \tag{4.4}$$

Hence $B^{(k)}$ is an isometry from $L^2(S^1)$ into the space

$$\mathcal{F} = \Bigg\{\psi(z) \text{ entire, } \psi(z+\omega a) = \mathrm{e}^{\mathrm{i}ak}\psi(z) \text{ and } \|\psi\|^2_{\mathcal{F}} = \frac{1}{2\pi\hbar}\int_0^a \mathrm{d}q\int_{-\infty}^{\infty}\mathrm{d}p\, \mathrm{e}^{-p^2/(\omega\hbar)}|\psi(z)|^2 < \infty, z = \omega q - \mathrm{i}p\Bigg\}. \tag{4.5}$$

We see that the space $\mathcal{F}$ is similar to the usual Fock space. Since $B^{(k)}$ also maps $L^2(S^1)$ onto $\mathcal{F}$, as we will see, we have a complete analogy with the standard case. Obviously, we can define the following orthonormal set in $\mathcal{F}$:

$$\{\psi_n(z) := (B^{(k)}|n; k\rangle)(z) = \langle z; k|n; k\rangle | n \in \mathbb{Z}\} \tag{4.6}$$

and it is not hard to compute the functions

$$\psi_n(z) = \left(\frac{4\pi\hbar}{a^2\omega}\right)^{1/4} \exp\left(-\frac{\hbar}{2\omega}\left(\frac{2\pi}{a}n + k\right)^2\right) \exp\left(\frac{\mathrm{i}}{\omega}\left(\frac{2\pi}{a}n + k\right)z\right). \tag{4.7}$$

To prove that $B^{(k)}$ is surjective is equivalent to proving that these functions form a basis for $\mathcal{F}$. But, after (4.7), this amounts to the existence of a Fourier series for any $\psi \in \mathcal{F}$, as is the case, i.e.

$$\psi(z) = \sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}(2\pi n/a + k)z/\omega} \qquad \forall\psi \in \mathcal{F} \tag{4.8}$$

because of the quasiperiodicity of the functions in $\mathcal{F}$ (we have introduced, for convenience, a factor $\mathrm{e}^{\mathrm{i}kz/\omega}$ in the usual Fourier series). Using (4.7) and the orthonormality of the set $\{\psi_n\}$, expression (4.8) becomes

$$\psi(z) = \sum_{n=-\infty}^{\infty} (\psi_n|\psi)_{\mathcal{F}}\psi_n(z) \qquad \forall\psi \in \mathcal{F} \tag{4.9}$$

where $(\cdot|\cdot)_{\mathcal{F}}$ denotes the inner product of $\mathcal{F}$ and

$$(\psi_n|\psi)_{\mathcal{F}} = a_n\left(\frac{a^2\omega}{4\pi\hbar}\right)^{1/4} \mathrm{e}^{\hbar(2\pi n/a+k)^2/(2\omega)}. \tag{4.10}$$

Clearly, there is a one-to-one correspondence between the coefficients $(\psi_n|\psi)_{\mathcal{F}}$ and a_n, so the set $\{\psi_n\}$ is a basis and $B^{(k)}$ is surjective.

We can write now some expressions for the inverse B^{-1} of $B^{(k)}$:

$$|B^{-1}\psi\rangle = \sum_{n=-\infty}^{\infty}(\psi_n|\psi)_{\mathcal{F}}|n; k\rangle \tag{4.11a}$$

$$= \frac{1}{2\pi\hbar}\int_0^a \mathrm{d}q\int_{-\infty}^{\infty}\mathrm{d}p\, \mathrm{e}^{-p^2/(\omega\hbar)}\psi(z)|z^*; k\rangle \qquad z = \omega q - \mathrm{i}p. \tag{4.11b}$$

5. Coherent states on the torus

All the results of the preceding sections can be easily generalized to a higher number of dimensions. With this purpose in mind, let us take a unitary basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ in $\mathbb{R}^n$ as well as a set of real numbers $\{a_1, a_2, \ldots, a_n\}$ and let us consider the associated lattice $\mathcal{L}$ [RS 78], that is,

$$\mathcal{L} = \left\{ \boldsymbol{a} \in \mathbb{R}^n | \boldsymbol{a} = \sum_{i=1}^{n} m_i a_i \boldsymbol{e}_i, m_i \in \mathbb{Z} \right\}. \tag{5.1}$$

In the same way, we define the dual basis $\{\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_n\}$ by $\boldsymbol{\epsilon}_i \cdot \boldsymbol{e}_j = \delta_{ij}$ and the dual lattice by

$$\mathcal{L}' = \left\{ \boldsymbol{b} \in \mathbb{R}^n | \boldsymbol{b} = \sum_{i=1}^{n} m_i \frac{2\pi}{a_i} \boldsymbol{\epsilon}_i, m_i \in \mathbb{Z} \right\}. \tag{5.2}$$

The corresponding basic cells $\mathbb{T}^n$ and $(\mathbb{T}^{n\prime})$ are n-dimensional tori,

$$\mathbb{T}^n = \left\{ \boldsymbol{q} \in \mathbb{R}^n | \boldsymbol{q} = \sum_{i=1}^{n} q_i \boldsymbol{e}_i, 0 \leqslant q_i < a_i \right\} \tag{5.3}$$

$$(\mathbb{T}^{n\prime}) = \left\{ \boldsymbol{k} \in \mathbb{R}^n | \boldsymbol{k} = \sum_{i=1}^{n} k_i \boldsymbol{\epsilon}_i, 0 \leqslant k_i < \frac{2\pi}{a_i} \right\}. \tag{5.4}$$

We shall define the n-dimensional WBZ transform T as a unitary map from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{T}^n \times (\mathbb{T}^{n\prime}))$ [JA 82, FO 89], given by

$$(T\psi)(\boldsymbol{q}, \boldsymbol{k}) = \sum_{\boldsymbol{a} \in \mathcal{L}} \mathrm{e}^{\mathrm{i}\boldsymbol{a}\cdot\boldsymbol{k}} \psi(\boldsymbol{q} - \boldsymbol{a}) \qquad \boldsymbol{z} \in \mathbb{T}^n \qquad \boldsymbol{w} \in (\mathbb{T}^{n\prime}) \tag{5.5}$$

with $\psi \in L^2(\mathbb{R}^n)$. The functions $T\psi$ verify

$$(T\psi)(\boldsymbol{q} + \boldsymbol{a}, \boldsymbol{k} + \boldsymbol{b}) = \mathrm{e}^{\mathrm{i}\boldsymbol{a}\cdot\boldsymbol{k}} (T\psi)(\boldsymbol{q}, \boldsymbol{k}) \qquad \boldsymbol{a} \in \mathcal{L}, \boldsymbol{b} \in \mathcal{L}'. \tag{5.6}$$

From now on, we shall fix a value of $\boldsymbol{k}$, so that expression (5.5) defines a projection $T^{(\boldsymbol{k})}$ onto $L^2(\mathbb{T}^n)$. We use the notation $T^{(\boldsymbol{k})}\psi = \psi^{(\boldsymbol{k})}$.

CS on the torus are obtained as the image under $T^{(\boldsymbol{k})}$ of the n–dimensional Weyl–Heisenberg CS $\eta_{\boldsymbol{q},\boldsymbol{p}} \in L^2(\mathbb{R}^n)$, which we write as

$$\eta_{\boldsymbol{q},\boldsymbol{p}}(\boldsymbol{x}) = \exp\left(\frac{\mathrm{i}}{\hbar} \boldsymbol{p} \cdot \left(\boldsymbol{x} - \frac{\boldsymbol{q}}{2} \right) \right) \eta_0(\boldsymbol{x} - \boldsymbol{q}) \tag{5.7}$$

where $\boldsymbol{x}, \boldsymbol{q}, \boldsymbol{p} \in \mathbb{R}^n$, and the fiducial state $\eta_0 \in L^2(\mathbb{R}^n)$ is chosen to be a normalized Gaussian:

$$\eta_0(\boldsymbol{x}) = \left(\frac{\omega}{\pi\hbar} \right)^{n/4} \exp\left(-\frac{\omega}{2\hbar} \boldsymbol{x}^2 \right). \tag{5.8}$$

In this case, the functions $\eta^{(\boldsymbol{k})}_{\boldsymbol{q},\boldsymbol{p}}$ take the form [GO 96]

$$\eta^{(\boldsymbol{k})}_{\boldsymbol{q},\boldsymbol{p}}(\boldsymbol{q}') = \left(\frac{\omega}{\pi\hbar} \right)^{n/4} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{q}/(2\hbar)} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{q}'/\hbar} \mathrm{e}^{-\omega(\boldsymbol{q}-\boldsymbol{q}')^2/(2\hbar)} \Theta\left(\frac{1}{2\hbar} \Delta[\hbar\boldsymbol{k} - \boldsymbol{p} + \mathrm{i}\omega G(\boldsymbol{q} - \boldsymbol{q}')] \middle| \Omega \right) \tag{5.9}$$

where G is the $n \times n$ symmetric matrix of the lattice, whose elements are $g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$; Δ is an $n \times n$ diagonal matrix with elements a_i; Ω is another $n \times n$ matrix given by

$$\Omega = \mathrm{i}\frac{\omega}{2\pi\hbar} \Delta G \Delta \tag{5.10}$$

and Θ is the n-dimensional Theta function [MU 83]:

$$\Theta(\boldsymbol{z}|\Omega) = \sum_{\boldsymbol{m}\in\mathbb{Z}^n} \exp(\mathrm{i}\pi \boldsymbol{m}\cdot\Omega\boldsymbol{m}) \exp(2\mathrm{i}\boldsymbol{m}\cdot\boldsymbol{z}). \tag{5.11}$$

We thus have the following n-dimensional version of theorem (3.1):

Theorem 5.1. For each $\boldsymbol{k} \in (\mathbb{T}^{n\prime})$, the family of functions

$$\{\eta^{(\boldsymbol{k})}_{\boldsymbol{q},\boldsymbol{p}} \equiv |\boldsymbol{q},\boldsymbol{p};\boldsymbol{k}\rangle | (\boldsymbol{q},\boldsymbol{p}) \in \mathbb{T}^n \times \mathbb{R}^n\} \tag{5.12}$$

given by (5.9), is a set of coherent states in $L^2(\mathbb{T}^n)$; i.e. they verify the resolution of unity:

$$\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{T}^n} \mathrm{d}\boldsymbol{q} \int_{R^n} \mathrm{d}\boldsymbol{p}\, |\boldsymbol{q},\boldsymbol{p};\boldsymbol{k}\rangle\langle \boldsymbol{q},\boldsymbol{p};\boldsymbol{k}| = I. \tag{5.13}$$

Most generalizations of the one-dimensional results are straightforward [GO 96]. Here we simply write the expression for the product $\langle \boldsymbol{q}',\boldsymbol{p}';\boldsymbol{k}|\boldsymbol{q},\boldsymbol{p};\boldsymbol{k}\rangle$. After a rather lengthy calculation, we obtain

$$\langle \boldsymbol{q}',\boldsymbol{p}';\boldsymbol{k}|\boldsymbol{q},\boldsymbol{p};\boldsymbol{k}\rangle = \frac{2^n}{\sqrt{g}A}\left(\frac{\pi\hbar}{\omega}\right)^{n/2} \mathrm{e}^{\mathrm{i}(\boldsymbol{p}\cdot\boldsymbol{q}-\boldsymbol{p}'\cdot\boldsymbol{q}')/(2\hbar)}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{q}'-\boldsymbol{q})}\mathrm{e}^{-[(\hbar\boldsymbol{k}-\boldsymbol{p}')^2+(\hbar\boldsymbol{k}-\boldsymbol{p})^2]/(2\omega\hbar)}$$
$$\times\Theta\left(\pi\Delta^{-1}\left[\boldsymbol{q}'-\boldsymbol{q}+\frac{\mathrm{i}}{\omega}G^{-1}(2\hbar\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{p}')\right]\middle|\Omega'\right) \tag{5.14}$$

where $g = \det G$, $A = a_1 a_2 \ldots a_n$ and $\Omega' = -2\Omega^{-1}$. Therefore, we also get

$$\langle \boldsymbol{q},\boldsymbol{p};\boldsymbol{k}|\boldsymbol{q},\boldsymbol{p};\boldsymbol{k}\rangle = \frac{2^n}{\sqrt{g}A}\left(\frac{\pi\hbar}{\omega}\right)^{n/2}\mathrm{e}^{-(\hbar\boldsymbol{k}-\boldsymbol{p})^2/(\omega\hbar)}\Theta\left(\mathrm{i}\frac{2\pi}{\omega}\Delta^{-1}G^{-1}(\hbar\boldsymbol{k}-\boldsymbol{p})\middle|\Omega'\right). \tag{5.15}$$

Finally, it can be shown [GO 96] that when the lattice is orthogonal, the CS $\eta^{(\boldsymbol{k})}_{\boldsymbol{q},\boldsymbol{p}}$ factorize out like a product of one-dimensional CS given by (3.4), i.e.

$$\eta^{(\boldsymbol{k})}_{\boldsymbol{q},\boldsymbol{p}}(\boldsymbol{q}') = \prod_{i=1}^{n} \eta^{(k_i)}_{q_i,p_i}(q'_i). \tag{5.16}$$

6. Physical properties of the CS on the circle

This section is devoted to the discussion of the physical properties of the CS on the circle introduced in section 3 (a complete study has been realized in [GO 96]). As these states have been constructed by decomposition of the standard Weyl–Heisenberg CS in $L^2(\mathbb{R})$, it seems to us that comparison between both cases could be illustrative. Moreover, it is known that the Weyl–Heisenberg CS have very nice quasiclassical properties, for instance, to minimalize the Heisenberg uncertainty relation, and it would be of great interest to reproduce such behaviour on the circle. As a matter of fact, we shall see that the physical properties of the CS on the circle depend mainly on some dimensionless parameter, related to the spread of the initial standard CS. If this spread is smaller than the length a of the circle, we get CS on the circle very similar to the standard CS. But if such a spread was comparable to or bigger than a, the CS on the circle are rather like plane waves.

We also discuss the relation between the CS parameters q, p and the expectation values in these states of the position and momentum operators on $L^2(S^1)$. Whereas for standard CS in $L^2(\mathbb{R})$ both things are the same, this is not the case on the circle. First we recall the correct definitions for the position and momentum operators on $L^2(S^1)$ (which show some significant differences from their analogues on the real line). Then, we will compute the expectation values of these operators and, finally, we devote some attention to the

Heisenberg uncertainty relation on the circle, but in a different and more suitable form than the usual one on the real line.

In order to provide an easier understanding of the somewhat complicated expressions, we illustrate our results with several figures. In any case, it has been possible to realize a complete analytic study [GO 96].

6.1. Quantum mechanics on the circle

The topology of the circle has important consequences for the quantum formalism on this configuration space. Indeed, experience shows that a direct translation of the formalism on the real line leads to serious inconsistencies [CN 68, ZA 69, LE 76]. For instance, it is known that the (angular) momentum operator on $L^2(S^1)$ has discrete spectrum. Moreover, functions in its domain must verify the constraint $\varphi(a) = \mathrm{e}^{\mathrm{i}ak}\varphi(0)$, where a is the length of the circle and $k \in [0, 2\pi/a)$ is a parameter as in section 2 [RS 75]. Thus, in fact, there is not one but a family of momentum operators on $L^2(S^1)$, labelled by k and which we denote by $P^{(k)}$. As a consequence, a canonical commutation relation as in $L^2(\mathbb{R})$

$$[Q, P^{(k)}] = \mathrm{i}\hbar \tag{6.1}$$

with position operator Q defined as usual, is inconsistent in $L^2(S^1)$. Heisenberg's uncertainty relation is even more troublesome, because of the compact spectrum of Q on $L^2(S^1)$. In effect, this relation allows the position dispersion to be bigger than a, which has no physical meaning.

All these problems can be solved choosing the unitary operator $E = \exp(\mathrm{i}2\pi Q/a)$ as a better representation for the position on the circle [LE 76]. It has precisely a circle as its spectrum and its commutator with the momentum operator is

$$[P^{(k)}, E] = \frac{2\pi\hbar}{a}E \tag{6.2}$$

which poses no domain problems. From this fundamental relation (6.2) we can also deduce an uncertainty relation more suitable for the circle [LE 76]. Since E is unitary but not self-adjoint, the dispersion ΔE should be defined in the form

$$(\Delta E)^2 := \langle E^\dagger E\rangle - |\langle E\rangle|^2 = 1 - |\langle E\rangle|^2 \tag{6.3}$$

so that relation (6.2) yields, by the usual method, the following Heisenberg uncertainty relation:

$$(\Delta P^{(k)})^2 \frac{(\Delta E)^2}{1-(\Delta E)^2} \geqslant \left(\frac{\pi}{a}\hbar\right)^2. \tag{6.4}$$

Note that now, when $\Delta P^{(k)} = 0$ we must have $\Delta E = 1$, which is a more appropriate result because of the compactness of the position variable on the circle. Moreover, relation (6.4) reduces to the usual Heisenberg uncertainty relation when $\Delta E \ll 1$ [LE 76]. Thus, we will call E the 'angle' operator and from now on we will use it as the quantum representation for the position on the circle.

6.2. Physical properties of the CS on the circle

6.2.1. Probability density. Let us begin the study of the basic physical properties of the CS $|q, p; k\rangle$ by computing its probability density $\mathcal{P}_{q,p;k}(q')$. The wavefunction $\eta^{(k)}_{q,p}(q')$ of these states is given by expression (3.4). As they are not normalized, the probability density will be

$$\mathcal{P}_{q,p;k}(q') = \frac{|\eta^{(k)}_{q,p}(q')|^2}{\langle q, p; k|q, p; k\rangle} \tag{6.5}$$

which, making use of expression (3.12), yields

$$\mathcal{P}_{q,p;k}(q') = \left(\frac{\omega}{\pi\overline{h}}\right)^{1/2} \mathrm{e}^{-\omega(q-q')^2/\overline{h}} \frac{|\theta\big(a[(k\overline{h} - p) + \mathrm{i}\omega(q - q')]/(2\overline{h});\, \rho_1\big)|^2}{\theta(a(k\overline{h} - p)/(2\overline{h});\, \rho_1^{1/2})}. \tag{6.6}$$

In order to clarify the notation, we introduce two new variables,

$$u := \frac{1}{a}(q' - q) \qquad v := \frac{a}{2\pi\overline{h}}(p - k\overline{h}) \tag{6.7}$$

as well as the dimensionless parameter

$$\alpha := \frac{a^2}{2\overline{h}}\omega. \tag{6.8}$$

In this way, the probability density, from now on denoted by $\mathcal{P}_\alpha(u; v)$, looks like

$$\mathcal{P}_\alpha(u; v) = \frac{1}{a}\sqrt{\frac{2\alpha}{\pi}}\mathrm{e}^{-2\alpha u^2} \frac{|\theta(\pi v + \mathrm{i}\alpha u;\, \mathrm{e}^{-\alpha})|^2}{\theta(\pi v;\, \mathrm{e}^{-\alpha/2})}. \tag{6.9}$$

This is a periodic function of u and v, in both cases with period 1. This corresponds to a period a for $q' - q$ and a period $2\pi\overline{h}/a$ for p. To give a general idea of its main properties, we show in figure 1 some significant cases. We observe that for high values of α (approximately $\alpha > 15$) the probability density is, with good accuracy, a Gaussian regardless of the value of v. That is, we have the same result as for the standard Weyl–Heisenberg CS. Note that, for these values of α, the width of the Gaussian is always smaller than a. On the other hand, for small values of α the probability density is no longer a Gaussian and its shape depends crucially on the value of v. Only when $v = \frac{1}{2}$ (i.e. $p = ((2n+1)\pi/a + k)\overline{h}$, with $n \in \mathbb{Z}$), does it look like a wavepacket for all values of α (right-hand side of the figure 1). In all the other cases it tends to be a plane wave when $\alpha \to 0$ (in the left-hand side of the figure 1 we show the case $v = 0$).

6.2.2. Expectation value of the angle operator. To compute the expectation value of E in the CS $|q, p; k\rangle$, we make use of the following relation

$$E|q, p; k\rangle = \mathrm{e}^{\mathrm{i}\pi q/a} \left|q, p + \frac{2\pi}{a}\overline{h}; k\right\rangle \tag{6.10}$$

which is easily deduced from the obvious action of E on the orthonormal basis $|n; k\rangle$ in $L^2(S^1)$ (see expression (3.6)),

$$E|n; k\rangle = |n + 1; k\rangle \qquad \forall n \in \mathbb{Z} \tag{6.11}$$

as well as from expression (3.8) for the coefficients of the CS $|q, p; k\rangle$ in this basis. We denote the expectation value of E by $\langle E\rangle(u, v)$, with v as in (6.7) but

$$u := \frac{q}{a} \tag{6.12}$$

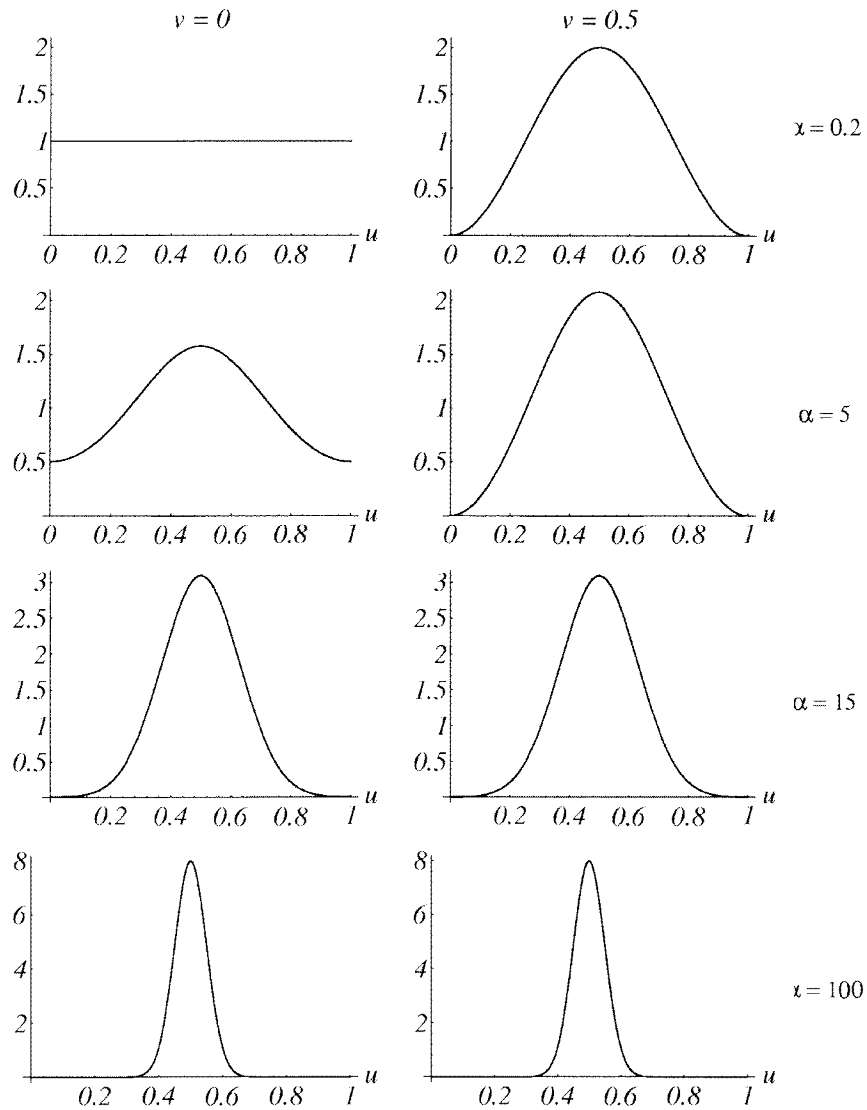

Figure 1. The functions $a\mathcal{P}_\alpha(u-\frac{1}{2},0)$ (left) and $a\mathcal{P}_\alpha(u-\frac{1}{2},\frac{1}{2})$ (right), for several values of α.

from now on. We also continue using the parameter α defined in (6.8). Taking together equations (6.10), (3.9), (3.11) and (3.12) we finally arrive at

$$\langle E\rangle(u,v)=\frac{\langle q,p;k|E|q,p;k\rangle}{\langle q,p;k|q,p;k\rangle}$$
$$=\mathrm{e}^{\mathrm{i}2\pi u}\mathrm{e}^{-\pi^2/(2\alpha)}\frac{\theta(\pi(v-\frac{1}{2});\mathrm{e}^{-\alpha/2})}{\theta(\pi v;\mathrm{e}^{-\alpha/2})}. \tag{6.13}$$

Of course, this is a periodic function of u but it is also an even periodic function of v with period 1. As all the factors excepting $\mathrm{e}^{\mathrm{i}2\pi u}$ are real positive [WW 27, ER 81, MU 83], we can write

$$\langle E\rangle(u,v)=\mathrm{e}^{\mathrm{i}2\pi u}|\langle E\rangle|(v). \tag{6.14}$$

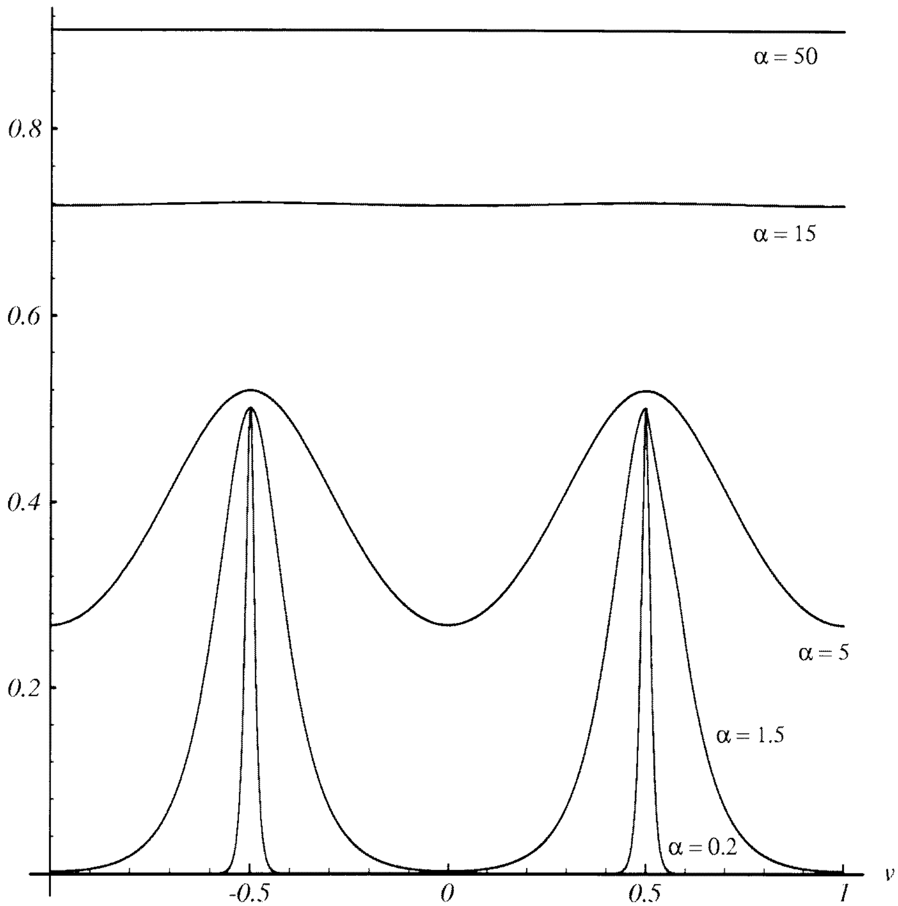

Figure 2. The function $|\langle E\rangle|(v)$, for several values of α.

We show the function $|\langle E\rangle|(v)$ in figure 2, for some values of α. Note that, in general, it is not possible to interpret the expectation value of E as a measure of the average position of the CS on the circle because of the dependence on v. However, observe in figure 2 that for high values of α, the function $|\langle E\rangle|(v)$ is almost constant. Thus in these cases we get the usual interpretation of the CS parameter q as the average position of the quantum state. On the other hand, we have seen in figure 1 that for small values of α most of the CS are nearly plane waves, hence it is not so important that the average position in these states cannot be well defined.

6.2.3. Expectation value of the momentum operator. We begin the calculation by observing that the vectors of the basis $|n;k\rangle$ in $L^2(S^1)$ (see expression (3.6)) are eigenvectors of the momentum operator $P^{(k)}$,

$$P^{(k)}|n;k\rangle = \hbar\left(\frac{2\pi}{a}n + k\right)|n;k\rangle. \tag{6.15}$$

Thus, we can write

$$\langle q, p; k|P^{(k)}|q, p; k\rangle = \hbar \sum_{n=-\infty}^{\infty} \left(\frac{2\pi}{a}n + k\right) |c_n^{(q,p;k)}|^2 \tag{6.16}$$

where the coefficients $c_n^{(q,p;k)}$ are given by expressions (3.8) and (3.2). Also using the formulae (3.11) and (3.12) we finally find

$$\langle P^{(k)}\rangle(p) = \frac{\langle q, p; k|P^{(k)}|q, p; k\rangle}{\langle q, p; k|q, p; k\rangle} = p + \frac{\hbar\alpha}{2a}\frac{\theta'(\pi v; e^{-\alpha/2})}{\theta(\pi v; e^{-\alpha/2})} \tag{6.17}$$

where, for the sake of clarity, we use the two variables p, v at the same time, and

$$\theta'(z; \rho) = \frac{d}{dz}\theta(z; \rho) = 2i \sum_{n=-\infty}^{\infty} n\rho^{n^2} e^{2inz} \qquad |\rho| < 1. \tag{6.18}$$

It is interesting to note that when $v = n/2$, with $n \in \mathbb{Z}$, i.e. $p = (n\pi/a + k)\hbar$, expression (6.17) reduces to $\langle P^{(k)}\rangle(p) = p$ as in the standard CS case. For other values of v, the difference between $\langle P^{(k)}\rangle$ and p depends on the parameter α. To show this, let us first rewrite equation (6.17) using only the variable v:

$$\langle P^{(k)}\rangle(v) = \frac{2\pi\hbar}{a}\left(v + \frac{\alpha}{4\pi}\frac{\theta'(\pi v; e^{-\alpha/2})}{\theta(\pi v; e^{-\alpha/2})}\right) + k\hbar. \tag{6.19}$$

We represent the function $(a/2\pi\hbar)\left(\langle P^{(k)}\rangle(v) - k\hbar\right)$ in figure 3, for some values of α (remember that $2\pi\hbar/a$ is the 'natural unit' for p). We see that for high values of α, the CS parameter p is a good approximation for the expectation value of the momentum operator. But for small values of α, this expectation value tends to take some discrete values for almost all values of v [GO 96]. These are the 'plane-wave' states of figure 1.

6.2.4. Heisenberg uncertainty relation. We conclude the study of the basic physical properties of the CS $|q, p; k\rangle$ with some comments about the Heisenberg uncertainty relation for these states. In the following we try to verify whether some of the CS $|q, p; k\rangle$ minimalize relation (6.4), which has to be used on the circle, as we remarked above.

Let us denote by $\Delta^{(k)}_{(q,p)}A$ the dispersion of an operator A in the CS $|q, p; k\rangle$. We begin by computing this dispersion for the angle operator E. According to expression (6.3) we get

$$(\Delta^{(k)}_{(q,p)}E)^2 = 1 - |\langle E\rangle(u, v)|^2 = 1 - |\langle E\rangle|(v)^2 \tag{6.20}$$

where $|\langle E\rangle|(v)$ can be obtained from expression (6.13).

On the other hand, the dispersion $\Delta^{(k)}_{(q,p)}P^{(k)}$ of the momentum operator requires a few more calculations. First, we have to compute the expectation value $\langle (P^{(k)})^2\rangle$, which after (6.15) can be written as

$$\langle (P^{(k)})^2\rangle(p) = \frac{\hbar^2}{\langle q, p; k|q, p; k\rangle} \sum_{n=-\infty}^{\infty} \left(\frac{2\pi}{a}n + k\right)^2 |c_n^{(q,p;k)}|^2. \tag{6.21}$$

Hence, again making use of formulae (3.8), (3.2), (3.11) and (3.12) we get, after a rather lengthy but straightforward calculation,

$$\langle (P^{(k)})^2\rangle(p) = \left(\frac{\hbar\alpha}{2a}\right)^2 \frac{\theta''(\pi v; e^{-\alpha/2})}{\theta(\pi v; e^{-\alpha/2})} + p\left(\frac{\hbar\alpha}{a}\frac{\theta'(\pi v; e^{-\alpha/2})}{\theta(\pi v; e^{-\alpha/2})} + p\right) + \frac{\hbar^2\alpha}{a^2} \tag{6.22}$$

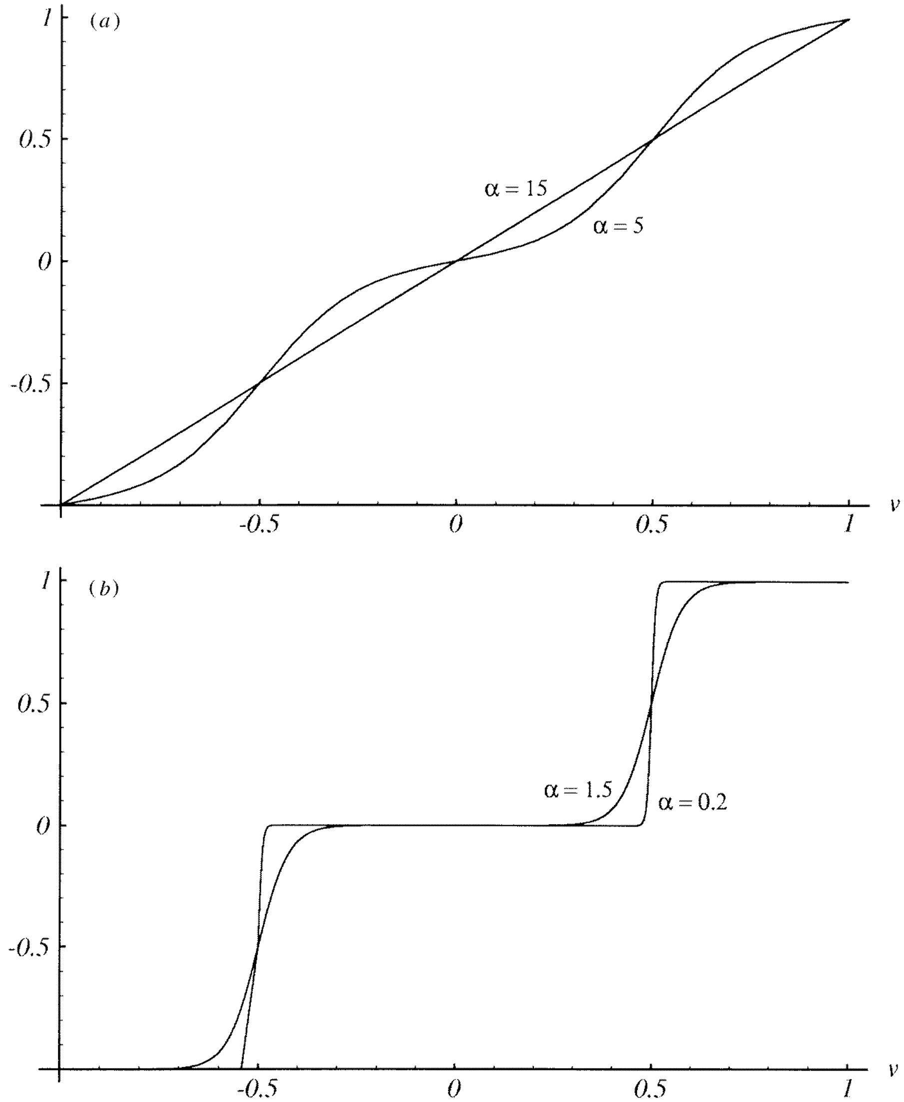

Figure 3. The function $\frac{a}{2\pi\hbar}(\langle P^{(k)}\rangle(v) - k\hbar)$, for several values of α.

with $\theta''(z;\rho) = \mathrm{d}^2\theta(z;\rho)/\mathrm{d}z^2$. Finally, equations (6.22) and (6.17) taken together yield

$$(\Delta^{(k)}_{(q,p)}P^{(k)})^2 = \left(\frac{\hbar}{a}\right)^2\left[\frac{\alpha^2}{4}\left(\frac{\theta''(\pi v;\mathrm{e}^{-\alpha/2})}{\theta(\pi v;\mathrm{e}^{-\alpha/2})} - \frac{\theta'(\pi v;\mathrm{e}^{-\alpha/2})^2}{\theta(\pi v;\mathrm{e}^{-\alpha/2})^2}\right) + \alpha\right]. \tag{6.23}$$

We are now able to discuss the uncertainty relation (6.4) for the CS $|q, p; k\rangle$. First, we define the uncertainty function

$$\Delta(v) := \frac{a}{2\pi}\frac{\Delta^{(k)}_{(q,p)}E}{\sqrt{1-(\Delta^{(k)}_{(q,p)}E)^2}}\Delta^{(k)}_{(q,p)}P^{(k)}$$

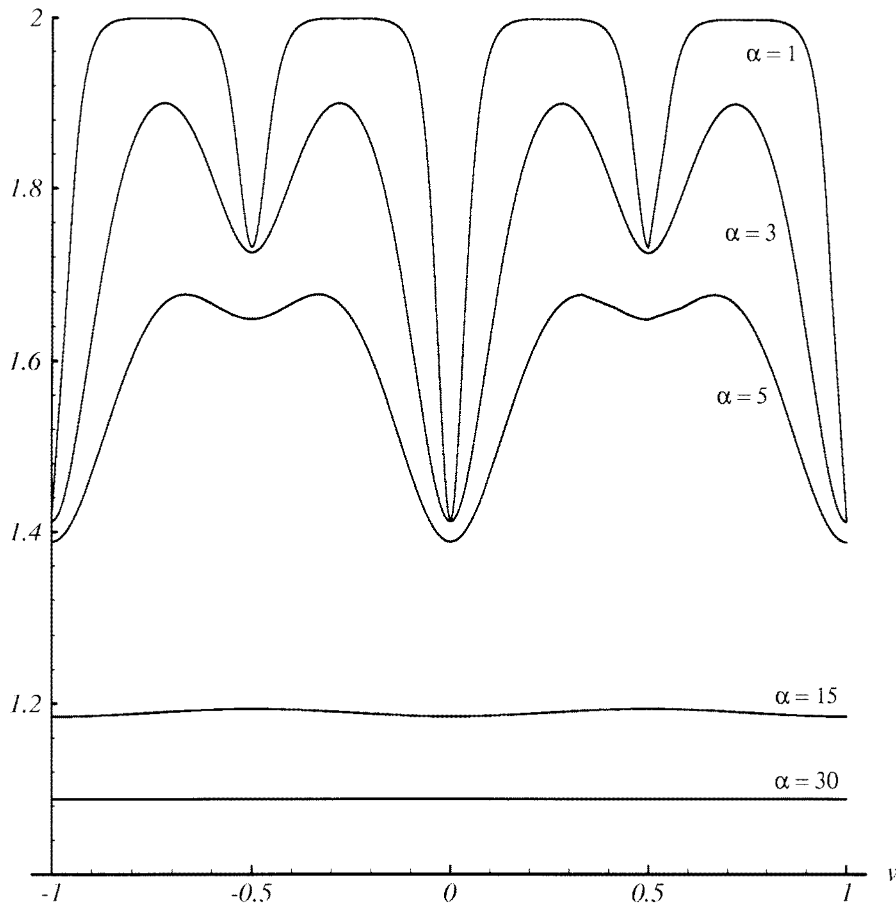

Figure 4. The function $\frac{2}{\hbar}\Delta(v)$ for several values of α.

$$= \frac{a}{2\pi}\left(\frac{1}{|\langle E\rangle|(v)^2} - 1\right)^{1/2} \Delta^{(k)}_{(q,p)} P^{(k)}. \tag{6.24}$$

In this way, relation (6.4) reduces to

$$\Delta(v) \geqslant \frac{\hbar}{2} \tag{6.25}$$

which looks more like a standard Heisenberg uncertainty relation, thus making this discussion more intuitive. In view of expressions (6.13) and (6.23) we arrive at the following formula:

$$\Delta(v)^2 = \left(\frac{\hbar}{2\pi}\right)^2 \alpha \left[\mathrm{e}^{\pi^2/\alpha}\frac{\theta(\pi v;\, \mathrm{e}^{-\alpha/2})^2}{\theta(\pi(v-\frac{1}{2});\, \mathrm{e}^{-\alpha/2})^2} - 1\right] \times \left[\frac{\alpha}{4}\left(\frac{\theta''(\pi v;\, \mathrm{e}^{-\alpha/2})}{\theta(\pi v;\, \mathrm{e}^{-\alpha/2})} - \frac{\theta'(\pi v;\, \mathrm{e}^{-\alpha/2})^2}{\theta(\pi v;\, \mathrm{e}^{-\alpha/2})^2}\right) + 1\right]. \tag{6.26}$$

We represent the function $(2/\hbar)\Delta(v)$ in figure 4, for some values of α. Observe its somewhat curious appearance. We remark that variable v is related to the CS parameter p, and that parameter α measures whether or not the CS $|q, p; k\rangle$ is similar to a standard CS. In figure 4, the value 1 on the vertical scale corresponds to a minimum uncertainty state, and in fact we see that for high values of α the function $\Delta(v)$ tends to this minimum, regardless of the value of v [GO 96]. Nevertheless, none of the CS $|q, p; k\rangle$ are real minimum uncertainty states, although we can obtain states as close to this limit as we wish, taking α sufficiently high.

In contrast, when the value of α is small we can see that the behaviour of the uncertainty relation for the CS $|q, p; k\rangle$ depends on the particular value of p, i.e. v. As $\Delta(v)$ is an even periodic function of v, we just need to consider the values $0 \leqslant v \leqslant \frac{1}{2}$. Thus, it can be proven [GO 96] that

$$\lim_{\alpha\to 0} \Delta(v) = \begin{cases} \frac{\sqrt{2}}{2}\hbar & \text{if } v = 0 \qquad \text{i.e. } p = \left(2n\frac{\pi}{a} + k\right)\hbar \qquad n \in \mathbb{Z} \\ \frac{\sqrt{3}}{2}\hbar & \text{if } v = \frac{1}{2} \qquad \text{i.e. } p = \left((2n+1)\frac{\pi}{a} + k\right)\hbar \qquad n \in \mathbb{Z} \\ \hbar & \text{in any other case.} \end{cases} \tag{6.27}$$

In other words, the uncertainty function Δ is bounded above at worst, by $\hbar$! Hence, we conclude that for the whole family of CS $|q, p; k\rangle$ we have

$$\hbar > \Delta(v) > \frac{\hbar}{2} \tag{6.28}$$

which, although strictly speaking does not correspond to minimum uncertainty states, shows a quite good behaviour of the CS $|q, p; k\rangle$ in this matter. The best behaviour is obtained for those states associated to the value $v = 0$.

7. Conclusions

As mentioned in the introduction, a family of CS on the circle was introduced in [KR 96]. These new CS are a particular case of the CS studied here. The authors of [KR 96] have not realized this fact and, moreover, they write in the introduction: '... The coherent states thus obtained are different from those defined in this paper [DG 93]. Nevertheless, it seems to us that the approach presented herein is a better one'. These CS are defined as

$$|\xi\rangle = \sum_j \xi^{-j} \mathrm{e}^{-j^2/2} |j\rangle \tag{7.1}$$

where $\xi = \mathrm{e}^{-l+\mathrm{i}\phi}$, $l \in \mathbb{R}$, $\phi \in S^1$ and $|j\rangle$ are the eigenvectors of the angular momentum operator. Two cases are considered in [KR 96]: the boson case when j takes integer values, and the fermion case when j takes half-integer values.

In the following, we prove that these CS (7.1) are particular cases of our CS $|z^*; k\rangle$. We have (see section 4)

$$|z^*; k\rangle = \sum_{n=-\infty}^{\infty} \psi_n(z)^* |n; k\rangle \tag{7.2}$$

where $z = \omega q - \mathrm{i}p$, and after (4.7)

$$\psi_n(z)^* = \left(\frac{4\pi\hbar}{a^2\omega}\right)^{1/4} \exp\left(-\frac{\hbar}{2\omega}\left(\frac{2\pi}{a}n + k\right)^2\right) \exp\left(-\frac{\mathrm{i}}{\omega}(\frac{2\pi}{a}n + k)z^*\right). \tag{7.3}$$

By analogy with [KR 96], from now on we set $\hbar = 1$ and $a = 2\pi$, so that $k \in [0, 1)$. If we also put $\xi = \exp(\mathrm{i}z^*/\omega) = \exp(-p/\omega + \mathrm{i}q)$, then (7.2) finally becomes

$$|z^*; k\rangle = \left(\frac{1}{\pi\omega}\right)^{1/4} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-(n+k)^2/(2\omega)} \xi^{-(n+k)} |n; k\rangle. \tag{7.4}$$

Now, simply comparing expressions (7.1) and (7.4) we see that both coincide (up to a constant factor) if we set $\omega = 1$ and $k = 0$ for the boson case, or $k = \pi/a = \frac{1}{2}$ for the fermion case. Indeed, for $k = 0$ we get

$$|z^*; 0\rangle = \left(\frac{1}{\pi}\right)^{1/4} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-n^2/2} \xi^{-n} |n; 0\rangle \tag{7.5}$$

which obviously coincides with (7.1) when j takes integer values, because expression (6.15) shows that $|n; 0\rangle$ are the boson eigenvectors of the angular momentum operator. In the same way, for $k = \frac{1}{2}$ we get

$$|z^*; \tfrac{1}{2}\rangle = \left(\frac{1}{\pi}\right)^{1/4} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-(n+1/2)^2/2} \xi^{-(n+1/2)} |n; \tfrac{1}{2}\rangle \tag{7.6}$$

which also equals (7.1) when j takes half-integer values, since $|n; \frac{1}{2}\rangle$ are now the fermion eigenvectors of the angular momentum operator, as we can see in expression (6.15). This ends the proof of our statement.

From the study of the physical properties of these CS we can state that they are very similar to the Heisenberg CS on $\mathbb{R}$, provided that the wideness of the wavefunction is small in comparison with the length of the configuration space S^1. Otherwise, the properties of these CS drastically depend on the values of p. Moreover, all the physical properties have a periodic behaviour in terms of p.

It is worthwhile to note that our CS are 'quasiminimal', i.e. although they do not minimize the Heisenberg uncertainty relation, the product of the dispersions of the angle and momentum operators is bounded above by $\hbar$.

Finally, we mention that these CS may be used to quantize the cylinder by means of Weyl correspondence [DG 92, DG 93, GO 96]. Work in this direction is in progress and the results will be published elsewhere.

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References

[AA 91] Ali S T, Antoine J-P and Gazeau J-P 1991 *Ann. Inst. H. Poincaré* **55** 829
Ali S T, Antoine J-P and Gazeau J-P 1991 *Ann. Inst. H. Poincaré* **55** 857
[AS 72] Abramowitz M and Stegun I A (ed) 1972 *Handbook of Mathematical Functions* (New York: Dover)
[BA 61] Bargmann V 1961 *Commun. Pure Appl. Math.* **14** 187
[BB 96] Bouzouina A and de Bièvre S 1996 *Commun. Math. Phys.* **178** 83
[CN 68] Carruthers P and Nieto M M 1968 *Rev. Mod. Phys.* **40** 411
[DB 89] de Bièvre S 1989 *J. Math. Phys.* **30** 1401
[DG 92] de Bièvre S and González J A 1992 Semiclassical behaviour of the Weyl correspondence on the circle *Group Theoretical Methods in Physics* vol I (Madrid: Ciemat)
[DG 93] de Bièvre S and González J A 1993 Semiclassical behaviour of coherent states on the circle *Quantization and Coherent States Methods* (Singapore: World Scientific)
[ER 81] Erdélyi A (ed) 1981 *Higher Transcendental Functions* vol II (Malabar, FL: Krieger)
[FO 89] Folland G B 1989 *Harmonic Analysis in Phase Space* (Princeton, NJ: Princeton University Press)
[GO 96] González J A 1996 Coherent states and quantization on the cylinder *PhD Thesis* Universidad de Valladolid
[IK 91] Isham C J and Klauder J R 1991 *J. Math. Phys.* **32** 607

[JA 82] Janssen A J E M 1982 *J. Math. Phys.* **23** 720

[KR 96] Kowalski K, Rembieliński J and Papaloucas L C 1996 *J. Phys. A: Math. Gen.* **29** 4149

[KS 85] Klauder J R and Skagerstam B-S 1985 *Coherent States: Applications in Physics and Mathematical Physics* (Singapore: World Scientific)

[LE 76] Lévy-Leblond J-M 1976 *Ann. Phys.* **101** 319

[MU 83] Mumford D 1983 *Tata Lectures on Theta* vol I (Boston, MA: Birkhäuser)

[PE 86] Perelomov A M 1986 *Generalized Coherent Sstates and their Applications* (Berlin: Springer)

[RS 75] Reed M and Simon B 1975 *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-adjointness* (New York: Academic)

[RS 78] Reed M and Simon B 1978 *Methods of Modern Mathematical Physics vol IV: Analysis of Operators* (New York: Academic)

[WW 27] Whittaker E T and Watson G N 1927 *A Course of Modern Analysis* (Cambridge: Cambridge University Press)

[ZA 68] Zak J 1968 *Phys. Rev.* **168** 686

[ZA 69] Zak J 1969 *Phys. Rev.* **187** 1803